Sound transmission in slowly varying circular and annular lined ducts with flow

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Sound transmission through straight circular ducts with a uniform inviscid mean flow and a constant acoustic lining (impedance wall) is classically described by a modal expansion. A natural extension for ducts with axially slowly varying properties (diameter and mean flow, wall impedance) is a multiple-scales solution. It is shown in the present paper that a consistent approximation of boundary condition and isentropic mean flow allows the multiple-scales problem to have an exact solution. Since the calculational complexities are no greater than for the classical straight duct model, the present solution provides an attractive alternative to a full numerical solution if diameter variation is relevant. A unique feature of the present solution is that it provides a systematic approximation to the hollow-to-annular cylinder transition problem in the turbofan engine inlet duct.

1. Introduction

The theory of sound propagation in straight ducts with constant impedance type boundary conditions and a homogeneous stationary medium is classical and well-established (Morse & Ingard 1968; Pierce 1981). At frequency ω, the sound field, satisfying the Helmholtz equation $(\nabla^2 + k^2)\phi = 0$, may be built up by superposition of eigensolutions or modes. These are certain shape-preserving fundamental solutions. The existence of these modes is a consequence of the relatively simple geometry, allowing separation of variables.

For cylindrical ducts, with associated cylindrical coordinate system (x, r, θ) the modes are given, in the usual complex notation, by exponentials and Bessel functions: $N J_m(\alpha r) \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}m\theta - \mathrm{i}kx}$ for a simple cylinder, and $[N J_m(\alpha r) + M Y_m(\alpha r)] \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}m\theta - \mathrm{i}kx}$ for an annular cylinder.

The eigenvalue m, or circumferential wavenumber, is, due to the periodicity in θ, an integer; the eigenvalue α, or radial wavenumber, is determined by the appropriate boundary condition at the duct wall(s), while the axial wavenumber k is related to α and ω via a dispersion relation. If we introduce a mean flow in the duct (motivated by aircraft turbofan engine applications, Nayfeh, Kaiser & Telionis 1975a, figure 1), the acoustic problem becomes rapidly much more difficult. Spatially varying mean flow velocities produce non-constant coefficients of the acoustic equations, which usually spoils the possibility of a modal expansion. Perhaps the simplest non-trivial mean flow is a uniform flow, in the limit of vanishing viscosity. Then modal solutions are possible, of a form rather similar to the one without flow.

A most important problem here is the way the sound field is transmitted through the vanishing mean flow boundary layer at the wall, which thus effectively modifies
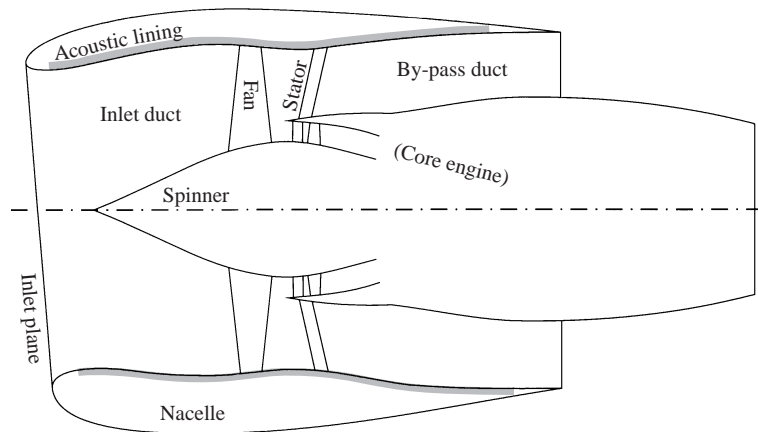
FIGURE 1. A turbofan engine.

the impedance boundary condition at the duct wall into an equivalent boundary condition in the limit to the duct wall. This modified boundary condition was first proposed by Ingard (1959) and later proved by Eversman & Beckemeyer (1972) and Tester (1973a) to be indeed the correct limit for a boundary layer which is much smaller than a typical acoustic wavelength.

In certain applications the geometry of a cylindrical duct is only an approximate model, and it is therefore of practical interest to consider sound transmission through ducts of varying cross-section. In general, this problem is, again, very difficult, and one usually resorts to numerical methods. However, quite often, especially when the duct carries a mean flow, the diameter variations of the duct are only gradual, thus introducing prospects of perturbation solutions. Indeed, several authors have utilized the small parameter related to the slow cross-section variations (Eisenberg & Kao 1971; Tam 1971; Huerre & Karamcheti 1973; Thompson & Sen 1984). A particularly interesting and systematic approach is the method of multiple scales elaborated by Nayfeh and co-workers, for both ducts without (Nayfeh & Telionis 1973) and with flow (Nayfeh, Telionis & Lekoudis 1975; Nayfeh, Kaiser & Telionis 1975b), and with hard and impedance walls. The multiple-scales technique provides a very natural generalization of modal solutions since a mode of a constant duct is now assumed to vary its shape according to the duct variations, in a way that amplitude and wavenumbers are slowly varying functions, rather than constants.

In Rienstra (1988) we proceeded along these lines, and presented an explicit multiple-scales solution of a problem similar to the one considered previously by Nayfeh *et al.* We considered a mode propagating in a slowly varying duct with impedance walls and containing almost uniform (inviscid, isentropic, irrotational) mean flow with vanishing boundary layer.

A somewhat puzzling aspect of Nayfeh *et al.*'s solutions was that without flow the differential equation for the slowly varying amplitude could be solved exactly, whereas with flow this was not the case. Also, in Rienstra (1985) the amplitude equation for a similar problem of a duct with (slowly varying) porous walls could be solved exactly. In Rienstra (1988) we showed that, at least in the present type of problem, an exact solution appears to be the rule rather than an exception, if the entire perturbation analysis is consistent at all levels. In the problem under consideration, Nayfeh *et al.* used an ad hoc mean flow velocity profile (quasi-one-dimensional with some assumed

boundary layer) which is not a solution of the mean flow equations, and, furthermore, in the case of a vanishing boundary layer they used an incorrect effective boundary condition, although at that time this was not known. Myers (1980) showed that Ingard's (1959) effective boundary condition for an impedance wall with uniform mean flow is to be modified significantly in the case of non-uniform mean flow along curved surfaces.

Both Myers' (1980) boundary condition and a consistent approximation of the mean flow is essential for the explicit solution presented.

In the present study we continue along these lines, and extend the theory to include an annular cylindrical geometry, in particular the transition from hollow to annular cylinder, and include some illustrative examples. These examples are taken from turbofan engine applications.

2. Formulation of the problem

We consider a cylindrical duct with slowly varying cross-section. Inside this duct we have a compressible inviscid perfect isentropic irrotational gas flow, consisting of a mean flow and acoustic perturbations. To the mean flow the duct is hard-walled, but for the acoustic field the duct is lined with an impedance wall.

It is convenient to make parameters dimensionless: spatial dimensions on a typical duct radius R_∞, densities on a reference value ρ_∞, velocities on a reference sound speed c_∞, time on R_∞/c_∞, pressure on $\rho_\infty c_\infty^2$, and velocity potential on $R_\infty c_\infty$. Note that the corresponding reference pressure p_∞ satisfies $\rho_\infty c_\infty^2 = \gamma p_\infty$, where γ is the (constant) ratio of specific heats at constant pressure and volume.

We then have in the cylindrical coordinates (x, r, θ), with unit vectors e_x, e_r and e_θ, the duct inner wall radius R_1 and outer wall radius R_2 given by

$$r = R_1(X), \ r = R_2(X), \ X = \varepsilon x, \ -\infty < x < \infty, \ 0 \leqslant \theta < 2\pi,$$

where ε is a small parameter, and $R_{1,2}$ is by assumption only dependent on ε through εx. As we will see ε is absent from the final results, but its rôle is necessary to legitimize and support the present systematic perturbation method. The fluid in the duct is described by (see, for example, Pierce 1981)

$$\tilde{\rho}_t + \nabla \cdot (\tilde{\rho}\tilde{v}) = 0, \tag{2.1a}$$

$$\tilde{\rho}(\tilde{v}_t + \tilde{v} \cdot \nabla \tilde{v}) + \nabla \tilde{p} = 0, \tag{2.1b}$$

$$\gamma \tilde{p} = \tilde{\rho}^\gamma, \quad \tilde{c}^2 = \frac{d\tilde{p}}{d\tilde{\rho}} = \tilde{\rho}^{\gamma-1} \tag{2.1c,d}$$

(with boundary and initial conditions), where $\tilde{v}$ is particle velocity, $\tilde{\rho}$ is density, $\tilde{p}$ is pressure, $\tilde{c}$ is sound speed (all dimensionless). Since we assumed the flow to be irrotational, we may introduce a velocity potential $\tilde{\phi}$, such that $\tilde{v} = \nabla \tilde{\phi}$. Using the vector identity $(\tilde{v} \cdot \nabla)\tilde{v} = \frac{1}{2}\nabla|\tilde{v}|^2 + (\nabla \times \tilde{v}) \times \tilde{v} = \frac{1}{2}\nabla|\tilde{v}|^2$, and the relation between $\tilde{p}$ and $\tilde{\rho}$, the above momentum equation may be integrated to a variant of Bernoulli's equation

$$\frac{\partial \tilde{\phi}}{\partial t} + \frac{1}{2}|\tilde{v}|^2 + \frac{c^2}{\gamma - 1} = \text{a constant.} \tag{2.2}$$

This flow is split up into a stationary (mean) flow part, and an acoustic perturbation. This acoustic part varies harmonically in time with circular frequency ω, and with small amplitude to allow linearization. To avoid a complicating coupling between the

two small parameters (ε and the acoustic amplitude), we assume this acoustic part much smaller than any relevant power of ε. In the usual complex notation (where the real part is assumed) we write then

$$\tilde{v} = V + v\mathrm{e}^{\mathrm{i}\omega t}, \quad \tilde{\phi} = \Phi + \phi\mathrm{e}^{\mathrm{i}\omega t}, \quad \tilde{\rho} = D + \rho\mathrm{e}^{\mathrm{i}\omega t}, \quad \tilde{p} = P + p\mathrm{e}^{\mathrm{i}\omega t}, \quad \tilde{c} = C + c\mathrm{e}^{\mathrm{i}\omega t}.$$

Substitution and linearization yields

the mean flow field

$$\nabla \cdot (D V) = 0, \tag{2.3a}$$

$$\tfrac{1}{2}|V|^2 + \frac{C^2}{\gamma - 1} = E \ \text{(a constant)}, \tag{2.3b}$$

$$C^2 = \gamma P/D = D^{\gamma - 1}; \tag{2.3c}$$

the acoustic field

$$\mathrm{i}\omega\rho + \nabla \cdot (D\nabla\phi + \rho V) = 0, \tag{2.4a}$$

$$\mathrm{i}\omega\phi + V \cdot \nabla\phi + \frac{p}{D} = 0, \tag{2.4b}$$

$$p = C^2\rho, \quad c = \tfrac{1}{2}(\gamma - 1)D^{-(\gamma+1)/2}p. \tag{2.4c,d}$$

The integration constant in the integrated momentum equation may be absorbed by ϕ. For the mean flow the duct wall is solid, so the normal velocity vanishes

$$V \cdot n_i = 0 \quad \text{at} \ \ r = R_i(X) \ (i = 1, 2), \tag{2.5}$$

where the outward-directed normal vectors at the wall are given by

$$n_1 = -\frac{e_r - \varepsilon R_1' e_x}{(1 + \varepsilon^2 R_1'^2)^{1/2}}, \quad n_2 = \frac{e_r - \varepsilon R_2' e_x}{(1 + \varepsilon^2 R_2'^2)^{1/2}}.$$

To define the mean flow an axial mass flux πF will be assumed such that the flow is subsonic everywhere. For the acoustic part the duct walls are locally reacting impedance walls with complex impedances $Z_1 = Z_1(X)$ and $Z_2 = Z_2(X)$ – slow variations of Z_i in X may be included – meaning that at the wall, at a hypothetical point with zero mean flow,

$$p = Z_i(v \cdot n_i).$$

However, this is not the boundary condition needed here. Since we deal with a fluid of vanishing viscosity, the boundary layer along the wall in which the mean flow tends to zero is of vanishing thickness, and we cannot apply a boundary condition at the wall. The required condition is for a point near the wall but still (just) inside the mean flow. For arbitrary mean flow along a (smoothly) curved wall it was given by Myers (1980, equation 15):

$$\mathrm{i}\omega(v \cdot n_i) = [\mathrm{i}\omega + V \cdot \nabla - n_i \cdot (n_i \cdot \nabla V)]\left(\frac{p}{Z_i}\right) \quad \text{at} \ \ r = R_i(X) \ (i = 1, 2) \tag{2.6}$$

while

$$p = 0 \quad \text{if} \ \ Z_i = 0.$$

Although included in the general formulation as a limiting case, it is easier to consider the solution for $Z = 0$ separately. For simplicity we will here only consider $Z_i \equiv 0$ as a special case.

The above equations and boundary conditions are evidently still insufficient to define a unique solution, and we need additional conditions for mean flow and sound

field. This will be done by assuming a certain behaviour. Since we are studying axial variations due to the geometry of the pipe, the natural choice is to consider a mean flow, almost uniform with axial variations only in X, and a sound field consisting of a constant-duct mode perturbed by the X-variations. Furthermore, this choice implies the absence of vorticity (apart from the vortex sheet along the wall), allowing the introduction of a velocity potential.

Before turning to the acoustic problem, we will derive in the next section the solution of the mean flow problem as a series expansion in ε. As noted before, a consistent mean flow expansion is necessary to obtain the explicit multiple-scale solution of the acoustic problem.

3. Mean flow

Since we assumed a mean flow, nearly uniform with axial variations in X only, we have

$$\boldsymbol{V} = U(X,r;\varepsilon)\boldsymbol{e}_x + V(X,r;\varepsilon)\boldsymbol{e}_r.$$

The cross-sectional mass flux is given by

$$2\pi \int_{R_1(X)}^{R_2(X)} D(X,r;\varepsilon)U(X,r;\varepsilon)r\,\mathrm{d}r = \pi F, \quad \text{a constant.} \tag{3.1}$$

Since the variations in x are through X only, we may assume the constants E and F to be independent of ε. Furthermore, by writing out the same mass equation (2.3a) in X and r, it follows that the small axial mass variations can only be balanced by small radial variations, so $V = O(\varepsilon)$, and hence

$$U(X,r;\varepsilon) = U_0(X) + O(\varepsilon^2), \quad V(X,r;\varepsilon) = \varepsilon V_1(X,r) + O(\varepsilon^3),$$

and so, with equations (2.3b) and (2.3c),

$$P(X,r;\varepsilon) = P_0(X) + O(\varepsilon^2), \quad D(X,r;\varepsilon) = D_0(X) + O(\varepsilon^2), \quad C(X,r;\varepsilon) = C_0(X) + O(\varepsilon^2).$$

From equation (3.1) it follows now immediately that

$$U_0(X) = \frac{F}{D_0(X)(R_2^2(X) - R_1^2(X))} \tag{3.2}$$

with D_0, P_0 and C_0 given by

$$\frac{1}{2}\left(\frac{F}{D_0(R_2^2 - R_1^2)}\right)^2 + \frac{1}{\gamma - 1}D_0^{\gamma-1} = E, \qquad P_0 = \frac{1}{\gamma}D_0^{\gamma}, \qquad C_0 = D_0^{(\gamma-1)/2}, \quad (3.3a\text{--}c)$$

where D_0 is to be determined numerically, per X. For V_1, we return to the continuity equation, which is to leading order

$$\frac{\partial}{\partial X}(D_0 U_0) + \frac{1}{r}\frac{\partial}{\partial r}(rD_0 V_1) = 0.$$

Under the boundary conditions

$$-\frac{\mathrm{d}R_i}{\mathrm{d}X}U_0 + V_1 = 0 \quad \text{at} \ \ r = R_i(X) \ (i = 1, 2)$$

(one of which is already satisfied through the application of (3.1) leading to (3.2)), we obtain the solution

$$V_1(X,r) = -\frac{F}{2rD_0(X)}\frac{\partial}{\partial X}\left(\frac{r^2 - R_1^2(X)}{R_2^2(X) - R_1^2(X)}\right). \tag{3.4}$$

The above solutions U_0, P_0, D_0 may be recognized as the well-known one-dimensional gas flow equations (e.g. Liepmann & Pluckett 1947). It should be stressed, however, that the radial velocity component V_1 is essential for a consistent mean flow description, and therefore necessary here.

4. The acoustic field

In this section we will derive the main result of the present paper: the explicit multiple-scales solution for a mode-like wave described by equations (2.4a–d) with (2.6). When we eliminate p and ρ we have the following differential equation and boundary conditions for ϕ:

$$\nabla \cdot (D\nabla\phi) - D\,(\mathrm{i}\omega + V \cdot \nabla)\left[\frac{1}{C^2}\,(\mathrm{i}\omega + V \cdot \nabla)\,\phi\right] = 0, \tag{4.1a}$$

$$\mathrm{i}\omega(\nabla\phi \cdot \boldsymbol{n}_i) = -(\mathrm{i}\omega + V \cdot \nabla - \boldsymbol{n}_i \cdot (\boldsymbol{n}_i \cdot \nabla V))\left[\frac{D}{Z_i}\,(\mathrm{i}\omega + V \cdot \nabla)\,\phi\right] \quad \text{at } r = R_i(X), \tag{4.1b}$$

and

$$(\mathrm{i}\omega + V \cdot \nabla)\,\phi = 0 \qquad \text{at} \quad r = R_i \quad \text{if} \quad Z_i = 0.$$

A straight-duct modal wave form would be a function of r multiplied by a complex exponential in θ and x. The mode-like wave we are looking for here is obtained by assuming the amplitude and axial and radial wave numbers to be slowly varying, i.e. depending on X (Nayfeh & Telionis 1973). So we assume

$$\phi(x, r, \theta; \varepsilon) = A(X, r; \varepsilon)\exp\left(-\mathrm{i}m\theta - \mathrm{i}\varepsilon^{-1}\int^X \mu(\xi)\,\mathrm{d}\xi\right). \tag{4.2}$$

Then the partial derivatives with respect to x become formally (suppressing the exponential)

$$\frac{\partial}{\partial x} = -\mathrm{i}\mu(X) + \varepsilon\frac{\partial}{\partial X},$$

$$\frac{\partial^2}{\partial x^2} = -\mu(X)^2 - \mathrm{i}\varepsilon\frac{\mathrm{d}\mu}{\mathrm{d}X} - 2\mathrm{i}\varepsilon\mu(X)\frac{\partial}{\partial X} + \varepsilon^2\frac{\partial^2}{\partial X^2}.$$

Substitution in (4.1a), and collecting like powers of ε yield up to order ε^2

$$D_0\mathcal{L}(A) = \frac{\mathrm{i}\varepsilon}{A}\left\{\frac{\partial}{\partial X}\left[\left(\frac{U_0\Omega}{C_0^2} + \mu\right)D_0A^2\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{V_1\Omega}{C_0^2}D_0A^2\right]\right\}, \tag{4.3}$$

where

$$\Omega = \omega - \mu U_0,$$

and the operator $\mathcal{L}$ is defined by

$$\mathcal{L} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\Omega^2}{C_0^2} - \mu^2 - \frac{m^2}{r^2}.$$

With $\boldsymbol{n}_i \cdot (\boldsymbol{n}_i \cdot \nabla V) = \varepsilon(\partial/\partial r)V_1 + O(\varepsilon^2)$, the boundary conditions (4.1b), up to order ε^2, are now

$$\mathrm{i}\omega\frac{\partial A}{\partial r} + \frac{\Omega^2 D_0 A}{Z_1} = \varepsilon\omega\mu\frac{\mathrm{d}R_1}{\mathrm{d}X}A + \frac{\mathrm{i}\varepsilon}{A}\left[U_0\frac{\partial}{\partial X} + V_1\frac{\partial}{\partial r} - \frac{\partial V_1}{\partial r}\right]\left(\frac{\Omega D_0 A^2}{Z_1}\right) \quad (r = R_1), \tag{4.4a}$$

$$i\omega\frac{\partial A}{\partial r} - \frac{\Omega^2 D_0 A}{Z_2} = \varepsilon\omega\mu\frac{\mathrm{d}R_2}{\mathrm{d}X}A - \frac{i\varepsilon}{A}\left[U_0\frac{\partial}{\partial X} + V_1\frac{\partial}{\partial r} - \frac{\partial V_1}{\partial r}\right]\left(\frac{\Omega D_0 A^2}{Z_2}\right) \quad (r = R_2), \quad (4.4b)$$

and

$$\left[i\Omega + \varepsilon\left(U_0\frac{\partial}{\partial X} + V_1\frac{\partial}{\partial r}\right)\right]A = 0 \quad \text{at} \quad r = R_i \quad \text{if} \quad Z_i = 0.$$

Now assume

$$A(X,r;\varepsilon) = A_0(X,r) + \varepsilon A_1(X,r) + \dots,$$

then substitution into equation (4.3) yields for the $O(1)$ and $O(\varepsilon)$ terms

$$\mathscr{L}(A_0) = 0, \tag{4.5a}$$

$$D_0\mathscr{L}(A_1) = \frac{i}{A_0}\left\{\frac{\partial}{\partial X}\left[\left(\frac{U_0\Omega}{C_0^2} + \mu\right)D_0 A_0^2\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{V_1\Omega}{C_0^2}D_0 A_0^2\right]\right\} \tag{4.5b}$$

with boundary conditions

$$i\omega\frac{\partial A_0}{\partial r} \pm \frac{\Omega^2 D_0 A_0}{Z_{1,2}} = 0, \quad (r = R_{1,2}) \tag{4.6a}$$

$$i\omega\frac{\partial A_1}{\partial r} \pm \frac{\Omega^2 D_0 A_1}{Z_{1,2}} = \omega\mu\frac{\mathrm{d}R_{1,2}}{\mathrm{d}X}A_0\dots$$

$$\pm\frac{i}{A_0}\left[U_0\frac{\partial}{\partial X} + V_1\frac{\partial}{\partial r} - \frac{\partial V_1}{\partial r}\right]\left(\frac{\Omega D_0 A_0^2}{Z_{1,2}}\right), \quad (r = R_{1,2}) \tag{4.6b}$$

The leading-order equation (4.5a) is, up to a radial coordinate stretching, Bessel's equation in r, with X acting only as a parameter. The mode-like solution we are looking for is then

$$A_0(X,r) = N(X)J_m(\alpha(X)r) + M(X)Y_m(\alpha(X)r), \tag{4.7}$$

where J_m and Y_m are the mth-order Bessel function of the first and second kind (Watson 1966). The corresponding boundary conditions (4.6a) produce the following equation for 'eigenvalue' α (continuous in X):

$$\frac{\alpha R_2 J_m'(\alpha R_2) - \zeta_2 J_m(\alpha R_2)}{\alpha R_2 Y_m'(\alpha R_2) - \zeta_2 Y_m(\alpha R_2)} = \frac{\alpha R_1 J_m'(\alpha R_1) + \zeta_1 J_m(\alpha R_1)}{\alpha R_1 Y_m'(\alpha R_1) + \zeta_1 Y_m(\alpha R_1)} = -\frac{M(X)}{N(X)}, \tag{4.8}$$

where

$$\zeta_1 = \frac{\Omega^2 D_0 R_1}{i\omega Z_1}, \quad \zeta_2 = \frac{\Omega^2 D_0 R_2}{i\omega Z_2}.$$

If $Z_i \equiv 0$ we have the hydrodynamic mode given by $\Omega = 0$, and the acoustic modes given by

$$\frac{J_m(\alpha R_2)}{Y_m(\alpha R_2)} = \frac{J_m(\alpha R_1)}{Y_m(\alpha R_1)} = -\frac{M(X)}{N(X)}.$$

Expression (4.8) itself is equal to $-M(X)/N(X)$, so only N is to be determined; α and μ are related by the dispersion relation

$$\alpha^2 + \mu^2 = \Omega^2/C_0^2.$$

It is convenient to introduce the reduced axial wavenumber

$$\sigma = \left(1 - (C_0^2 - U_0^2)\frac{\alpha^2}{\omega^2}\right)^{1/2}$$

which is μ scaled by ω and without the pure convection effects, so that

$$\mu = \omega \frac{C_0 \sigma - U_0}{C_0^2 - U_0^2}, \quad \frac{U_0 \Omega}{C_0^2} + \mu = \frac{\omega \sigma}{C_0}, \quad \Omega = \omega C_0 \frac{C_0 - U_0 \sigma}{C_0^2 - U_0^2}, \quad \text{and} \quad (C_0^2 - U_0^2)\frac{\alpha^2}{\omega^2} + \sigma^2 = 1.$$

The branch (i.e. sign) of σ is to be selected such that $\mathrm{Im}(\sigma) \leqslant 0$, $\mathrm{Re}(\sigma) \geqslant 0$ (quadrant IV) if the mode is propagating in the positive direction, and $\mathrm{Im}(\sigma) \geqslant 0$, $\mathrm{Re}(\sigma) \leqslant 0$ (quadrant II) if the mode is propagating in the negative direction. A single exception is to be made if impedance, frequency, and mean flow are such that the vortex sheet between the mean flow and impedance wall becomes (Helmholtz) unstable, corresponding to a σ in either quadrant I or III (Rienstra 1986; Tester 1973*b*; Koch & Möhring 1983; and others). This includes the border case of the hydrodynamic mode $\Omega = 0$ if $Z_i \equiv 0$. Although in principle included in the present results, we will not consider these cases here in detail.

Note that in the cylindrical duct case, with $R_1 = 0$, we have just $M(X) = 0$ so that

$$A_0(X,r) = N(X)J_m(\alpha(X)r),$$

and α is determined from

$$\alpha R_2 J_m'(\alpha R_2) - \zeta_2 J_m(\alpha R_2) = 0. \tag{4.9}$$

The amplitude functions $N(X)$ and $M(X)$ are determined from the condition that there exists a solution A_1. This is not trivial since we assumed the solution to behave in a certain way, namely to depend on X rather than x. Now suppose that we proceed and solve the equation for A_1, and subsequently find the necessary forms of N and M, then it would appear that we end up with similarly undetermined functions in A_1. So this approach looks rather inefficient. Indeed, it is not necessary to work out the equations for A_1 in detail. We only need a solvability condition (Nayfeh 1981), sufficient to yield the required equation for N.

Since the operator $r\mathscr{L}$ is self-adjoint in r, we have

$$\int_{R_1}^{R_2} A_0 \mathscr{L}(A_1) r \, \mathrm{d}r = R_2 \left[A_0 \frac{\partial A_1}{\partial r} - A_1 \frac{\partial A_0}{\partial r} \right]_{r=R_2} - R_1 \left[A_0 \frac{\partial A_1}{\partial r} - A_1 \frac{\partial A_0}{\partial r} \right]_{r=R_1}.$$

Further evaluation of this expression (using 4.6*b*) and the corresponding right-hand side of (4.5*b*) gives finally, after some calculation, the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}X} \left[D_0 \frac{\omega \sigma}{C_0} \int_{R_1}^{R_2} A_0^2(X,r) r \, \mathrm{d}r + \frac{D_0 U_0}{\Omega} \left(\zeta_2 A_0^2(X,R_2) + \zeta_1 A_0^2(X,R_1) \right) \right] = 0. \tag{4.10}$$

Use is made of equations (3.2) and (3.4), and the identities

$$\int^{R(X)} \frac{\partial}{\partial X} f(X,r) \, \mathrm{d}r = \frac{\mathrm{d}}{\mathrm{d}X} \int^{R(X)} f(X,r) \, \mathrm{d}r - \frac{\mathrm{d}R}{\mathrm{d}X} f(X,R)$$

and

$$U_0 \frac{\partial}{\partial X} + V_1 \frac{\partial}{\partial r} = U_0 \frac{\mathrm{d}}{\mathrm{d}X} \quad \text{along} \quad r = R(X).$$

The above equation (4.10) can be integrated immediately, with a constant of integration Q_0^2. This constant is determined by the initial amplitude of the mode entering the duct. Since the solution is linear, it is irrelevant here. The integral of $A_0^2 r$, finally to be evaluated, is a well-known integral of Bessel functions (see the Appendix), with

the result (using (4.8))

$$\int_{R_1}^{R_2} A_0^2(X,r)r\,\mathrm{d}r = \tfrac{1}{2}R_2^2\left(1 - \frac{m^2-\zeta_2^2}{\alpha^2 R_2^2}\right)A_0^2(X,R_2) - \tfrac{1}{2}R_1^2\left(1 - \frac{m^2-\zeta_1^2}{\alpha^2 R_1^2}\right)A_0^2(X,R_1).$$

Using the relations

$$A_0(X,R_1) = \frac{(2/\pi)N(X)}{\alpha R_1 Y_m'(\alpha R_1) + \zeta_1 Y_m(\alpha R_1)}, \quad A_0(X,R_2) = \frac{(2/\pi)N(X)}{\alpha R_2 Y_m'(\alpha R_2) - \zeta_2 Y_m(\alpha R_2)}$$

and some further simplifications we thus obtain the following expression for $N(X)$ (the principal result of the present paper):

$$\left(\frac{\tfrac{1}{2}\pi Q_0}{N}\right)^2 = \frac{\dfrac{D_0\omega\sigma R_2^2}{2C_0}\left(1 - \dfrac{m^2-\zeta_2^2}{\alpha^2 R_2^2}\right) + \dfrac{D_0 U_0}{\Omega}\zeta_2}{\left(\alpha R_2 Y_m'(\alpha R_2) - \zeta_2 Y_m(\alpha R_2)\right)^2} - \frac{\dfrac{D_0\omega\sigma R_1^2}{2C_0}\left(1 - \dfrac{m^2-\zeta_1^2}{\alpha^2 R_1^2}\right) - \dfrac{D_0 U_0}{\Omega}\zeta_1}{\left(\alpha R_1 Y_m'(\alpha R_1) + \zeta_1 Y_m(\alpha R_1)\right)^2}.$$
(4.11)

An interesting special case is the hard-walled duct, where $Z_i = \infty$, $\zeta_i = 0$. Then we have a real α and

$$\left(\frac{\tfrac{1}{2}\pi Q_0}{N}\right)^2 = \frac{D_0\omega\sigma}{2C_0}\left(\frac{R_2^2 - m^2/\alpha^2}{[\alpha R_2 Y_m'(\alpha R_2)]^2} - \frac{R_1^2 - m^2/\alpha^2}{[\alpha R_1 Y_m'(\alpha R_1)]^2}\right)$$
(4.12)

which is for the plane wave mode $m = 0$, $\alpha = 0$, $\sigma = 1$, given by

$$\left(\frac{Q_0}{N}\right)^2 = \frac{D_0\omega}{2C_0}(R_2^2 - R_1^2),$$

while $M(X) = 0$. Another special case is $Z_i = 0$. It is included in the present formulas as the limit $\zeta_i \to \infty$, but probably easier is to repeat the analysis with $Z_i = 0$ right from the start. We obtain then

$$\left(\frac{\tfrac{1}{2}\pi Q_0}{N}\right)^2 = \frac{D_0\omega\sigma}{2C_0}\left(\frac{1}{Y_m(\alpha R_2)^2} - \frac{1}{Y_m(\alpha R_1)^2}\right).$$
(4.13)

For a hollow cylinder, without an inner wall, the above general result (4.11) reduces, in the limit $R_1 \to 0$, to

$$\left(\frac{Q_0}{N}\right)^2 = \left(\frac{D_0\omega\sigma R_2^2}{2C_0}\left(1 - \frac{m^2-\zeta_2^2}{\alpha^2 R_2^2}\right) + \frac{D_0 U_0}{\Omega}\zeta_2\right)J_m(\alpha R_2)^2.$$
(4.14)

So the present solution is equally valid for hollow and annular cylindrical ducts, and hence includes the unique feature that it provides (apparently for the first time) a systematic approximation to the hollow-to-annular cylinder transition problem in the turbofan engine duct inlet. This aspect will be illustrated in the next section by an example.

If convenient, we may observe that for a hard-walled hollow cylinder the combination αR_2 is a constant, independent of X, so we can absorb some constant factors into Q_0 to obtain

$$\left(\frac{Q_0}{N}\right)^2 = \frac{D_0\omega\sigma R_2^2}{C_0}.$$
(4.15)

Of course, with a transition from a hollow to annular cylinder this is not advisable, because then it is required that we deal with the same Q_0.
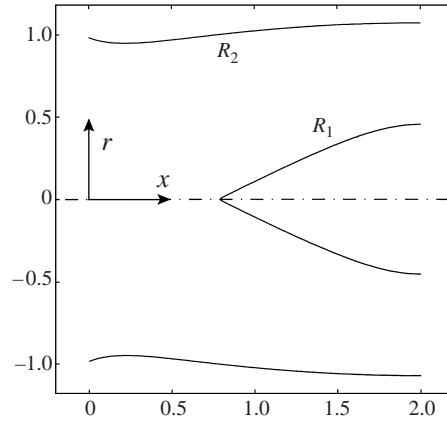
FIGURE 2. Sketch of geometry: engine inlet with spinner.

5. Example

In this section we will discuss an example of the previous theory. As noted above, in the results obtained the small parameter ε is not explicitly present anymore, so for convenience we will now return from the slow variable X to the physical (dimensionless) variable x.

A lined inlet duct of a CFM56-inspired turbofan engine, from inlet plane via (hard-walled) spinner to the inlet rotor plane, is given by (see figure 2)

$$R_2(x) = 1.073 - 0.198(1 - x/2)^2 + 0.109\exp(-11x/2),$$
$$R_1(x) = \max(0, 0.689 - \left[0.055 + 1.131(1 - x/2)^2\right]^{1/2}), \qquad 0 \leqslant x \leqslant 2.$$

The inlet plane is at about $x = 0$, the plane of narrowest duct diameter $R_2 = 0.949$ is at $x = 0.223$, the plane where $R_2 = 1$ is at $x = 0.773$ (the reference radius R_∞), the spinner top is at $x = 0.782$, and the fan plane is at $x = 2$, where $R_2 = 1.073$ and $R_1 = 0.454$. The impedances used are $Z_2 = 2 - \mathrm{i}$ and $Z_1 = \infty$. The ratio of specific heats is $\gamma = 1.4202$. Assuming the dimensionless density $D = 1$ far upstream, its value slightly below 1 at the inlet plane, and the inlet Mach number ~ 0.6, we choose $F = 0.559$ and $E = 2.514$. Density and Mach number are given in figure 3. A rotor blade number of 26, and a rotor tip Mach number slightly below 1 are taken, such that we would have (with a clean in-flow) rotor-alone noise of frequency $\omega = 25$ and $m = 26$, which is near cut-off. Just for the example we include a well cut-on second harmonic of $\omega = 50$ and $m = 26$, the origin of which is for the moment unimportant. The first radial left-running mode is considered, together (for comparison) with its right-running companion.

We start with $\omega = 25$. In figure 4(*a–c*) the radial wavenumber α, the axial wavenumber μ, and the reduced axial wavenumber σ are shown in the complex plane, parametrically varying with the duct position x. Initial positions are indicated by an open circle, intermediate positions by filled small circles. To be sure that we are looking at the same left- and right-running mode, both are found first, at the initial position $x = 0$, for no-flow conditions ($F = 0$), when both modes coincide. Then the modes are traced for increasing F. This can be seen in figure 4(*a*), the plot for α: the thin dotted line is α at $x = 0$ for increasing F.
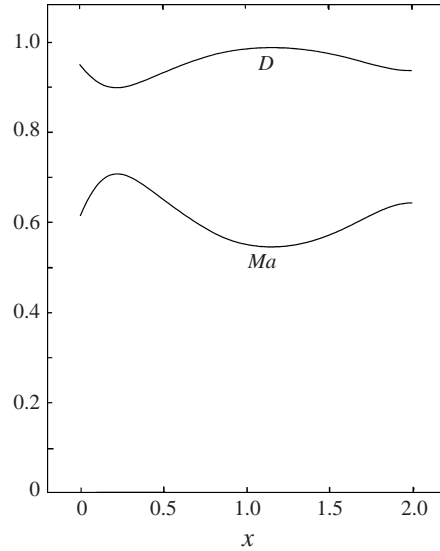
FIGURE 3. Dimensionless mean flow density and Mach number.

The cross-sectionally averaged amplitude functions A, given by

$$\bar{A}(X) = \left[\int_{R_1}^{R_2} |A(X,r)|^2 r \, dr \right]^{1/2},$$

are plotted in figure 5. The respective values are normalized to 1 at the beginning and end positions. Since it is of interest to measure the amount of dissipated acoustic energy, we introduce here the acoustic power $\mathscr{P}$ of a single mode through a duct cross-section. Following Goldstein (1976), we define the acoustic power at a surface S

$$\mathscr{P} = \int_S \boldsymbol{I} \cdot \boldsymbol{n}_S \, ds,$$

where $\boldsymbol{I}$ is the time-averaged acoustic intensity or energy flux, here given by

$$\boldsymbol{I} = \tfrac{1}{2} \operatorname{Re}[(p/D + \nabla\Phi \cdot \nabla\phi)(D\nabla\phi + \rho\nabla\Phi)^*],$$

with $*$ denoting the complex conjugate. Considering here for S a duct cross-section, we need the axial component of $\boldsymbol{I}$, which is to leading order

$$I_x = \frac{D_0\omega^2}{2C_0} \operatorname{Re}(\sigma)|\phi|^2 = \frac{D_0\omega^2}{2C_0} \operatorname{Re}(\sigma)|A_0|^2 \exp\left(2\int^x \operatorname{Im}\mu(\varepsilon\xi)\,d\xi\right), \qquad (5.1)$$

so that

$$\mathscr{P} = 2\pi \frac{D_0\omega^2}{2C_0} \operatorname{Re}(\sigma) \int_{R_1}^{R_2} |A_0(X,r)|^2 r \, dr \exp\left(2\int^x \operatorname{Im}\mu(\varepsilon\xi)\,d\xi\right), \qquad (5.2)$$

where (see the Appendix)

$$\int_{R_1}^{R_2} |A_0(X,r)|^2 r \, dr = -\frac{|A_0(X,R_2)|^2 \operatorname{Im}(\zeta_2) + |A_0(X,R_1)|^2 \operatorname{Im}(\zeta_1)}{\operatorname{Im}(\alpha^2)},$$

which is, of course, equivalent to $(\bar{A})^2$. For hard-walled ducts ($Z_i = \infty$, $\zeta_i = 0$) all eigenvalues α are real. Then $\mathscr{P} = 0$ for cut-off modes ($\operatorname{Re}(\sigma) = 0$). For cut-on modes,
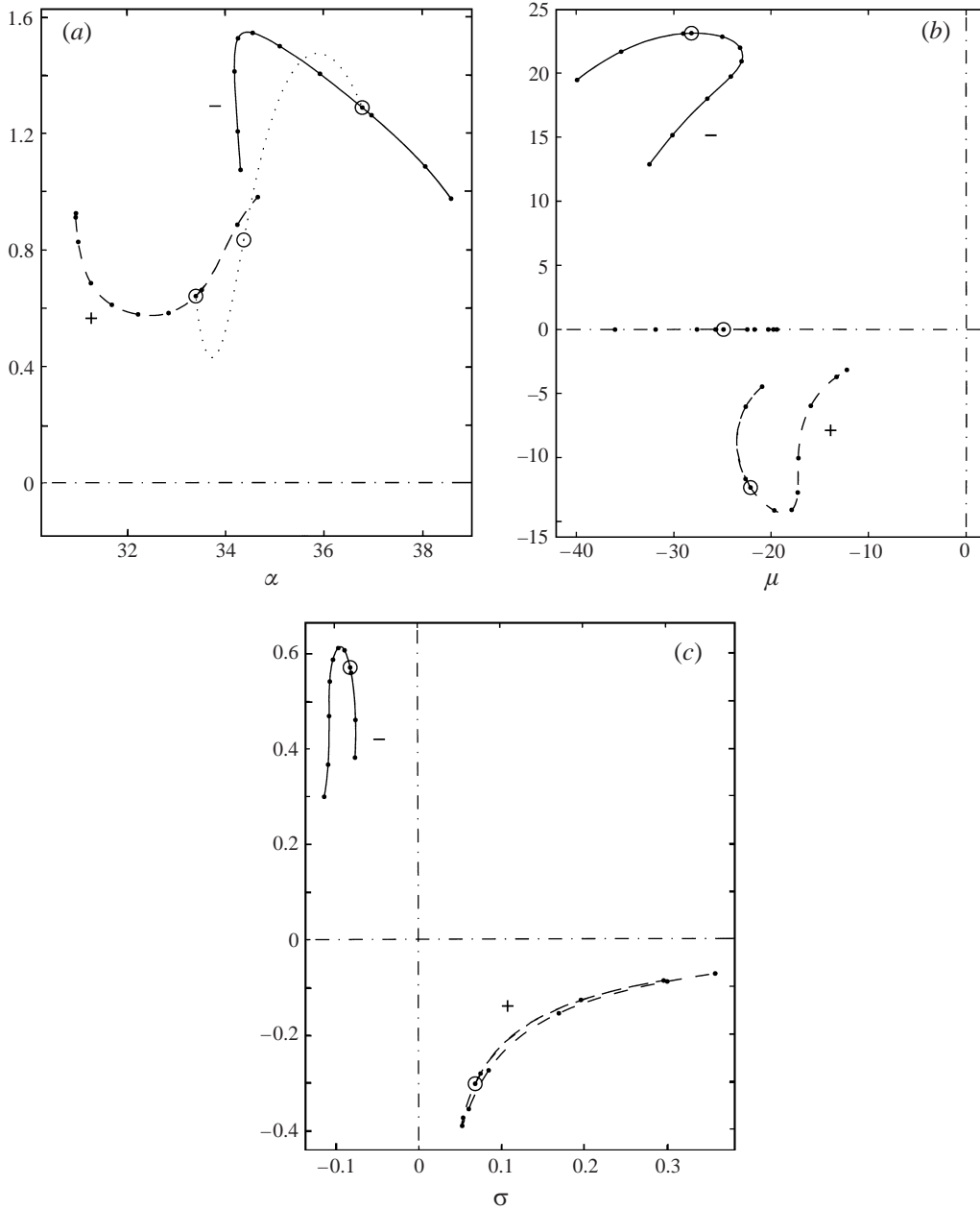
FIGURE 4. Complex-valued (*a*) radial wavenumber α_{m1}, (*b*) axial wavenumber μ_{m1} and (*c*) reduced wavenumber σ_{m1} ($\omega = 25$).

where σ is real, A_0 is also real and

$$\int_{R_1}^{R_2} A_0(X, r)^2 r \, dr = \frac{Q_0^2 C_0}{D_0 \omega \sigma}.$$

Since the value of $\mathscr{P}$ is highly dominated by the exponential part $\exp(2 \int \operatorname{Im}(\mu) \, d\xi)$, we have plotted the power both without (figure 6(*a*), linear scale) and with this exponent (figure 6(*b*), dB scale). Since for the majority of axial positions the left-running mode
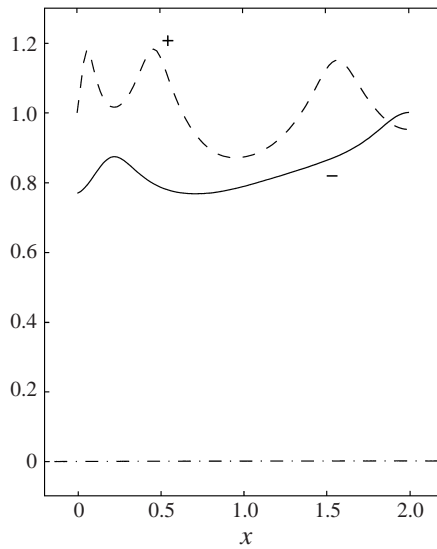
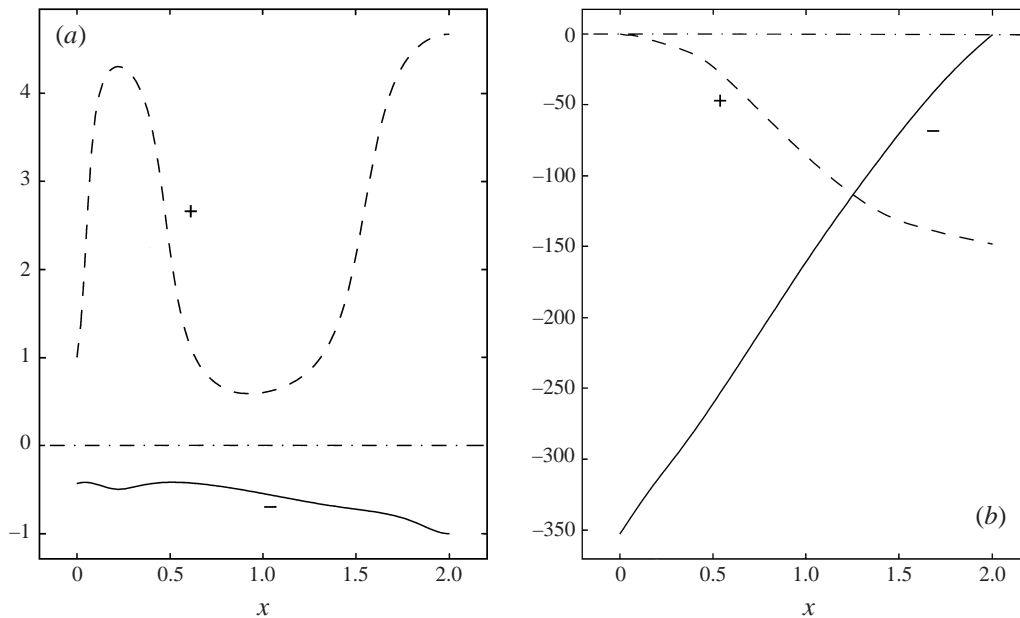FIGURE 5. Averaged amplitude $\bar{A}$ ($\omega = 25$).

FIGURE 6. Power (*a*) without exp-term (linear scale), (*b*) dB scale ($\omega = 25$).

was well cut-off, while the right-running one was not (figure 4*b,c*), the left-running mode is altogether very much more damped.

The acoustic pressure distribution inside the duct is displayed in figure 7. Iso-pressure contours are plotted of the field of the left-running mode. The pressure amplitude

$$|p(x,r)| = \left| -\mathrm{i} D_0 \Omega A_0 \exp\left(-\mathrm{i} \int^x \mu(\varepsilon\xi)\,\mathrm{d}\xi \right) \right|$$

is scaled such that the maximum is equal to 1. The plotted contour levels are in equal steps of $\frac{1}{10}$ between 0.1 and 1, supplemented with steps increasing in proportions of
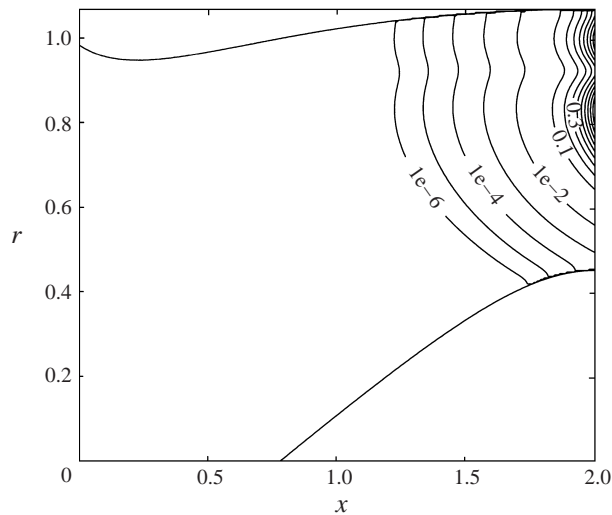
FIGURE 7. Iso-pressure contours ($\omega = 25$).

10 between 10^{-6} and 10^{-2} to reveal the lowest levels. The strong decay of this cut-off mode is very prominent.

For the well cut-on second harmonic $\omega = 50$ we have a similar sequence, but the effects are clearly less pronounced, as shown in figures 8–11. In spite of the considerable geometry variation, the averaged amplitude $\bar{A}$ is practically constant. Also, the power is now only dependent on its exponential part, i.e. the imaginary part of μ. This, in its turn, is only dependent on convection effects, as we can see from the nearly stationary reduced wavenumber σ. The upstream mode is now less damped than its downstream companion.

The iso-pressure contours of the left-running mode, plotted in figure 11, show some interesting features. The high-frequency wave fronts propagate almost parallel, like rays. The interaction with the liner is small, resulting in little damping, such that the iso-pressure lines are nearly parallel to the wall. The r^m-behaviour near the axis $r = 0$ creates in front of the spinner an area of very low sound pressure levels, leaving the spinner acoustically unimportant. Of course, the effect of the spinner is felt indirectly, via the mean flow. There is, for example, the local intensification of the field in the duct throat where the mean velocity is largest ($x \simeq 0.2$, $r \simeq 0.9$).

6. Discussion and conclusions

If the multiple-scales solution is valid, the mode-like wave behaves locally like a mode of a straight duct. Rotating with angular velocity ω/m, it propagates in the axial direction with or without attenuation (unattenuated or cut on: $\text{Im}(\mu) = 0$; attenuated or cut off: $\text{Im}(\mu) \neq 0$). The more interesting aspects here, are, of course, connected to the slow variations in X. These are mainly represented by the amplitude functions N, M and the phase function μ. When $R_{1,2}$ and $Z_{1,2}$ vary with X, the mode changes gradually, except at the points where the denominator of N (4.11) vanishes and the approximation breaks down. These points are just found at the double eigenvalues, i.e. where two eigenvalues μ (or α) coalesce. These are given by equation (4.8) and its partial derivative with respect to μ.

Clearly, the approximation breaks down because the two coalescing modes couple (the energy of the incident mode is distributed over the two) in a short region. A local
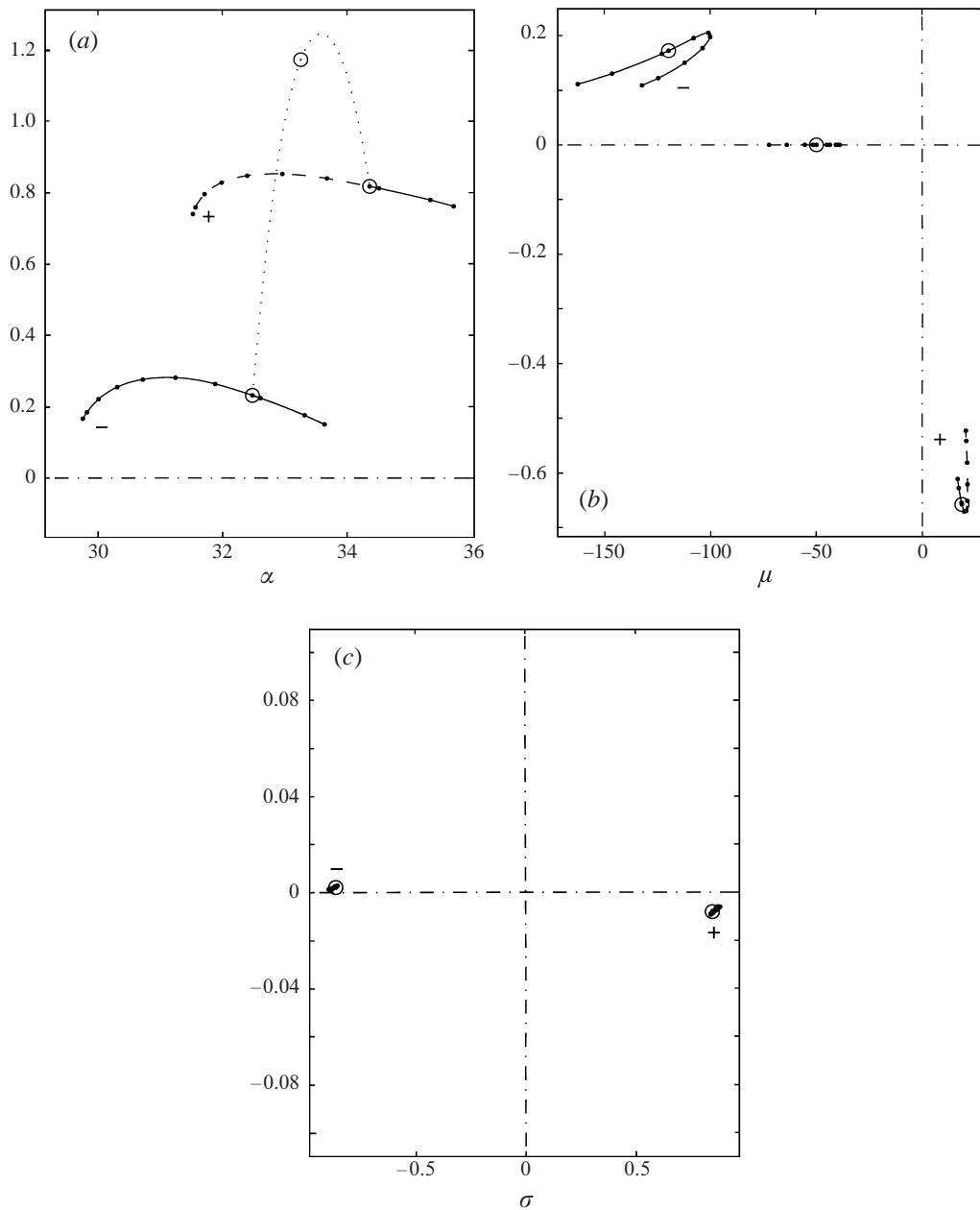
FIGURE 8. Complex-valued (*a*) radial wavenumber α_{m1}, (*b*) axial wavenumber μ_{m1}, and (*c*) reduced wavenumber σ_{m1}, ($\omega = 50$).

analysis, similar to what may be found in Nayfeh & Telionis (1973) for the no-flow hard-walled problem, is necessary to determine the resulting amplitudes.

In general the two modes propagate in the same direction but not necessarily. The second mode may run backwards while at the same time the incident mode becomes cut off in such a way that beyond that point no energy is propagated. Points with this behaviour are usually called turning points, since the incident mode is totally reflected
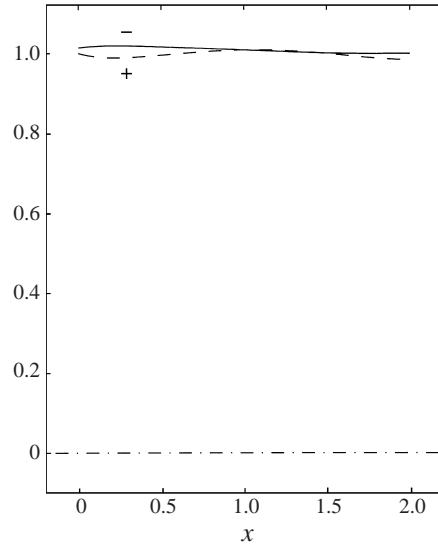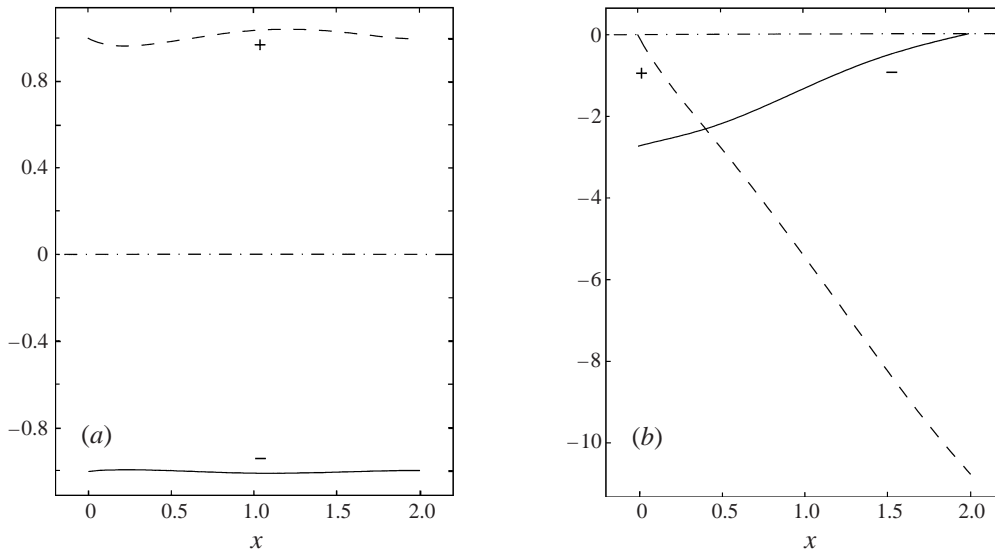
FIGURE 9. Averaged amplitude $\bar{A}$ ($\omega = 50$).

FIGURE 10. Power (*a*) without exp-term (linear scale), (*b*) dB scale ($\omega = 50$).

into the backward-running mode (we assume, of course, the absence of tunnelling by other interfering turning points).

Turning points occur in practice with hard-walled ducts ($Z = \infty$), where a real σ tends to zero to become pure imaginary (α is always real). At $\sigma = 0$, N is singular (4.12), (4.15), and the incident mode couples to a backwards-running mode. For real σ we have

$$\mathscr{P} \sim \mathrm{Re}(\sigma) \neq 0,$$

whereas for pure imaginary σ

$$\mathscr{P} = 0,$$

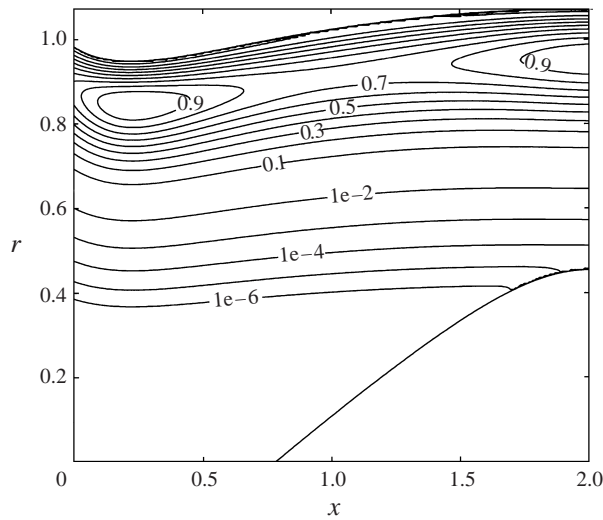so the mode must indeed reflect. Note that this behaviour is irrespective of the presence of mean flow.

FIGURE 11. Iso-pressure contours ($\omega = 50$).

In conclusion, we have found an explicit solution for the multiple-scale problem of modal sound propagation through slowly varying lined ducts with isentropic mean flow. It is shown that a consistent approximation of the boundary condition and mean flow allows the multiple-scales problem to have an exact solution.

Since the calculational complexities are no greater than for the classical straight duct model, the present solution provides an attractive alternative to a full numerical solution if diameter variation is relevant. The present solution is equally valid for hollow and annular cylindrical ducts, and hence includes the unique feature that it provides a systematic approximation to the hollow-to-annular cylinder transition problem in the turbofan engine duct inlet. This aspect is elaborated by an example.

The solution remains equally valid for hard-walled or no-flow ducts, but needs adaptation for a completely soft wall with $Z = 0$. The approximation breaks down at double eigenvalues when the mode couples with other modes. This occurs for example at cut-off points in a hard-walled duct.

Appendix.

Well-known integrals of Bessel functions (Watson 1966, p. 135) are

$$\int x \mathscr{C}_m(\alpha x) \tilde{\mathscr{C}}_m(\beta x)\, \mathrm{d}x = \frac{x}{\alpha^2 - \beta^2} \{ \beta \mathscr{C}_m(\alpha x) \tilde{\mathscr{C}}'_m(\beta x) - \alpha \mathscr{C}'_m(\alpha x) \tilde{\mathscr{C}}_m(\beta x) \},$$

$$\int x \mathscr{C}_m(\alpha x) \tilde{\mathscr{C}}_m(\alpha x)\, \mathrm{d}x = \tfrac{1}{2}(x^2 - m^2/\alpha^2) \mathscr{C}_m(\alpha x) \tilde{\mathscr{C}}_m(\alpha x) + \tfrac{1}{2} x^2 \mathscr{C}'_m(\alpha x) \tilde{\mathscr{C}}'_m(\alpha x),$$

where $\mathscr{C}_m$ and $\tilde{\mathscr{C}}_m$ are any linear combination of J_m and Y_m.

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